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Hyperbolic Conformal Geometry with Clifford Algebra[1]

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In this paper, we study hyperbolic conformal geometry following a Clifford algebraic approach. Similar to embedding an affine space into a one-dimensional higher linear space, we embed the hyperboloid model of the hyperbolic n-space in $\mathcal{R}^{n,1}$ into $\mathcal{R}^{n+1,1}$. The model is convenient for the study of hyperbolic conformal properties. Besides investigating various properties of the model, we also study conformal transformations using their versor representations.

1. INTRODUCTION

Hyperbolic geometry is an important branch of mathematics and physics. Among various models of the hyperbolic n-space, the hyperboloid model, which identifies the space with a branch $\mathcal{H}^n$ of the set

$$\mathcal{D}^n = \{x \in \mathcal{R}^{n,1} \mid x \cdot x = -1\} \tag{1.1}$$

has the following features:

- The model is isotropic in that at every point of $\mathcal{H}^n$, the metric of the tangent space is the same.
- A hyperbolic line AB is the intersection of $\mathcal{H}^n$ with the plane determined by vectors A, B through the origin. When viewed from the origin, line AB can be identified with a projective line in $\mathcal{P}^n$. Similarly, a hyperbolic r-plane can be identified with a projective r-plane in $\mathcal{P}^n$. Here $0 \leq r \leq n-1$. The geometry of r-planes can therefore be studied within the framework of linear subspaces in $\mathcal{R}^{n,1}$
- The tangent direction of a line l at a point A is a vector orthogonal

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to vector A in the plane determined by l and the origin. The angle of two intersecting lines is the Euclidean angle of their tangent directions at the intersection. This is the conformal property of the model.

- Let p, q be two points and let $d(p, q)$ be their hyperbolic distance. Then $p \cdot q = -\cosh d(p, q)$. This helps to transform a geometric problem on distances into an algebraic problem on inner products.
- A generalized circle is either a hyperbolic circle, a horocycle, or a hypercycle (equidistant curve). A generalized circle is the intersection of $\mathcal{H}^n$ with an affine plane in $\mathcal{R}^{n,1}$. Similarly, a generalized r-sphere is the intersection of $\mathcal{H}^n$ with an affine $(r + 1)$-plane. So the geometry of generalized r-spheres can be studied within the framework of affine subspaces in $\mathcal{R}^{n,1}$.
- The hyperbolic isometries are the orthogonal transformations of $\mathcal{R}^{n,1}$ which leave $\mathcal{H}^n$ invariant. In particular, they are all linear transformations.
- The model is similar to that of the spherical n-space in $\mathcal{R}^{n+1}$.

These features imply that we can use Clifford algebra, in particular, the version formulated in Hestenes and Sobczyk (1984), to study the hyperboloid model. A study on the two-dimensional case is carried out in Li (1997).

Similar to studying an affine n-space by embedding it into a linear $(n + 1)$-space as a hyperplane, we can study hyperbolic conformed properties of the hyperboloid model by embedding it into the null cone of the space $\mathcal{R}^{n+1,1}$. This model of hyperbolic n-space is called the homogeneous model in Li *et al.* (2000). It simplifies the study of hyperbolic conformal geometry.

Moreover, the homogeneous model provides a universal algebraic model for three geometries: Euclidean, spherical, and hyperbolic ones. In the universal model, different geometries correspond to different geometric interpretations of the same algebraic representation and the transfer from one geometry to another is realized by rescaling null vectors.

In this paper, we first review the hyperboloid model, then study properties of the homogeneous model. We also study conformal transformations using their versor representations. We use the terminology on hyperbolic geometry in Iversen (1992), Ratcliffe (1994), and Li (1997) and the notation for Clifford algebra as found in Hestenes and Sobczyk (1984).

2. THE HYPERBOLOID MODEL

For hyperbolic conformal geometry, the space $\mathcal{H}^n$ is not big enough, and we need its double covering space $\mathcal{D}^n$ defined by (1.1), called the double-hyperbolic n-space, or the hyperboloid model of the double-hyperbolic n-space. It has two branches, denoted by $\mathcal{H}^n$ and $-\mathcal{H}^n$, respectively.

Definition 2.1. An oriented generalized point of $\mathcal{D}_n$ is either a point, an end, or a direction. A point is an element in $\mathcal{D}^n$. An end (or oriented point at infinity) is a null half 1-space of $\mathcal{R}^{n,1}$. A direction (or oriented imaginary point) is a Euclidean half 1-space of $\mathcal{R}^{n,1}$. A point at infinity is a null 1-space of $\mathcal{R}^{n,1}$.

Algebraically, any oriented generalized point can be represented by a vector in $\mathcal{R}^{n,1}$; two vectors represent the same oriented generalized point if and only if they differ by a positive scale. A point at infinity is represented by a null vector, and two null vectors represent the same point at infinity if and only if they differ by a nonzero scale.

Let c be a point at infinity, and let p, q be points. Then p, q are on the same branch of $\mathcal{D}^n$ if and only if $c \cdot pc \cdot q > 0$.

Definition 2.2. 1. Let c be an end, $p \in \mathcal{H}^n$. If $c \cdot p < 0$, c is called an end of the branch $\mathcal{H}^n$; otherwise it is called an end of the branch $-\mathcal{H}^n$.

2. An r-plane of $\mathcal{D}^n$ is the intersection of $\mathcal{D}^n$ with an $(r + 1)$-space of $\mathcal{R}^{n,1}$.

3. The sphere at infinity of $\mathcal{D}^n$ is the whole set of points at infinity. An r-sphere at infinity of $\mathcal{D}^n$ is the intersection of the sphere at infinity with an $(r + 1)$-plane of $\mathcal{D}^n$, also called the sphere at infinity of the $(r + 1)$-plane.

In $\mathcal{G}_{n,1}$, an r-plane is represented by an $(r + 1)$-blade corresponding to the $(r + 1)$-space containing the r-plane. An $(n - 1)$-plane is called a hyperplane.

Definition 2.3. A generalized sphere is either a sphere, horosphere, or a hypersphere. It is determined by a pair (c, ρ), where c is a vector in $\mathcal{R}^{n,1}$ representing an oriented generalized point, called the center of the generalized sphere, and $\rho > 0$ is the generalized radius.

1. When c is a point, the set $\{p \in \mathcal{D}^n | p \cdot c = -(1 + \rho)\}$ is called a sphere. It is on the same branch with point c.

2. When c is an end, the set $\{p \in \mathcal{D}^n | p \cdot c = -\rho\}$ is called a horosphere. It is on the same branch with the end c.

3. When c is a direction, the set $\{p \in \mathcal{D}^n | p \cdot c = -\rho\}$ is called a hypersphere. The hyperplane of $\mathcal{D}^n$ orthogonal to c is called the axis of the hypersphere.

Definition 2.4. 1. A generalized r-sphere is a generalized sphere in an $(r + 1)$-plane which is taken as a double-hyperbolic $(r + 1)$-space.

2. A total sphere of $\mathcal{D}^n$ is either a generalized sphere, a hyperplane, or the sphere at infinity. A total r-sphere is an r-dimensional generalized sphere, plane, or sphere at infinity.

3. THE HOMOGENEOUS MODEL

Let a_0 be a fixed vector in $\mathcal{R}^{n+1,1}$, $a_0^2 = 1$. The space represented by $a_0^{\sim}$ is Minkowski, denoted by $\mathcal{R}^{n,1}$. The mapping F: $x \mapsto x - a_0$ maps $\mathcal{D}^n$ in a one-to-one manner onto the set

$$\mathcal{N}_{a_0}^n = \{x \in \mathcal{R}^{n+1,1} | x \cdot x = 0, x \cdot a_0 = -1\} \tag{3.2}$$

The set $\mathcal{N}_{a_0}^n$, together with the mapping F, is called the homogeneous model of the double-hyperbolic n-space.

In this model, an end or point at infinity is represented by a null vector orthogonal to a_0; a direction is represented by a vector of unit square orthogonal to a_0.

3.1. Representation of Total Spheres

Let p, q be null vectors representing two points on the same branch of $\mathcal{D}^n$. Let $d(p, q)$ be the hyperbolic distance between the two points. Then $p \cdot q = 1 - \cosh d(p, q)$. By this equality and Definition 2.3, we get the following result.

Lemma 3.1. 1. A point p is on the sphere (c, ρ), when p, c are understood to be null vectors representing the points, if and only if $p \cdot c = -\rho$.

2. A point p is on the horosphere (or hypersphere) (c, ρ), where p is understood to be the null vector representing the point, if and only if $p \cdot c = -\rho$.

The following is the first main theorem on the homogeneous model.

Theorem 3.2. Let $B_{r-1,1}$ be a Minkoswski r-blade in $\mathcal{G}_{n+1,1}$, $2 \leq r \leq n + 1$. Then $B_{r-1,1}$ represents a total $(r - 2)$-sphere: a point represented by a null vector p is on the total $(r - 2)$-sphere if and only if $p \wedge B_{r-1,1} = 0$.

1. If $a_0 \cdot B_{r-1,1} = 0$, then $B_{r-1,1}$ represents an $(r - 2)$-sphere at infinity.
2. If $a_0 \cdot B_{r-1,1}$ is Euclidean, then $B_{r-1,1}$ represents an $(r - 2)$-sphere.
3. If $a_0 \cdot B_{r-1,1}$ is degenerate, then $B_{r-1,1}$ represents an $(r - 2)$-horosphere.
4. If $a_0 \cdot B_{r-1,1}$ is Minkowski, but $a_0 \wedge B_{r-1,1} \neq 0$, then $B_{r-1,1}$ represents an $(r - 2)$-hypersphere.
5. If $a_0 \wedge B_{r-1,1} = 0$, then $B_{r-1,1}$ represents an $(r - 2)$-plane.

When $r = n + 1$, the dual form of the above theorem is as follows:

Theorem 3.3. Let s be a vector of positive signature in $\mathcal{R}^{n+1,1}$; then $s^{\sim}$ represents a total sphere. A point represented by a null vector p is on the total sphere if and only if $p \cdot s = 0$.

1. If $a_0 \wedge s = 0$, then $s^{\sim}$ represents the sphere at infinity. The sphere at infinity is represented by $a_0^{\sim}$.
2. If $a_0 \wedge s$ is Minkowski, then $s^{\sim}$ represents a sphere. The sphere (c, ρ) is represented by $(c - \rho a_0)^{\sim}$, where c is a null vector representing the point.
3. If $a_0 \wedge s$ is degenerate, then $s^{\sim}$ represents a horosphere. The horosphere (c, ρ) is represented by $(c - \rho a_0)^{\sim}$.
4. If $a_0 \wedge s$ is Euclidean, but $a_0 \cdot s \neq 0$, then $s^{\sim}$ represents a hypersphere. The hypersphere (c, ρ) is represented by $(c - \rho a_0)^{\sim}$.
5. If $a_0 \cdot s = 0$, then $s^{\sim}$ represents a hyperplane. A hyperplane with normal direction c is represented by $c^{\sim}$.

Proof. We prove Theorem 3.3 only. Theorem 3.2 can be proved using Theorem 3.3.

1. If $a_0 \wedge s = 0$, by definition, any null vector in the space $s^{\sim}$ represents a point at infinity, and vice versa.

2. If $a_0 \wedge s$ is Minkowski, then $a_0 \cdot s \neq 0$. Let ϵ be the sign of $a_0 \cdot s$. Let

$$\mathbf{c} = -\epsilon \frac{P_{a_0}^{\perp}(s)}{|P_{a_0}^{\perp}(s)|}, \qquad \rho = \frac{|a_0 \cdot s|}{|a_0 \wedge s|} - 1 \tag{3.3}$$

Then $\mathbf{c}$ is a point, $\rho > 0$, as $|a_0 \wedge s|^2 = (a_0 \cdot s)^2 - s^2 < (a_0 \cdot s)^2$. Let $s' = -\epsilon s/|a_0 \wedge s|$. Then $s' = \mathbf{c} - (1 + \rho)a_0 = c - \rho a_0$, where $c = \mathbf{c} - a_0$. A point represented by a null vector p is on the sphere $(\mathbf{c}, \rho)$ if and only if $p \cdot s' = 0$, which is equivalent to $p \cdot s = 0$.

3. If $a_0 \wedge s$ is degenerate, then $|a_0 \cdot s| = |s| \neq 0$. Let ϵ be the sign of $a_0 \cdot s$. Let

$$c = -\epsilon P_{a_0}^{\perp}(s), \qquad \rho = |a_0 \cdot s| = |s| \tag{3.4}$$

Then c is an end, $\rho > 0$. Let $s' = -\epsilon s$. Then $s' = c - \rho a_0$. A point represented by a null vector p is on the horosphere (c, ρ) if and only if $p \cdot s = 0$.

4. If $a_0 \wedge s$ is Euclidean, but $a_0 \cdot s \neq 0$, let ϵ be the sign of $a_0 \cdot s$. Let

$$c = -\epsilon \frac{P_{a_0}^{\perp}(s)}{|P_{a_0}^{\perp}(s)|}, \qquad \rho = \frac{|a_0 \cdot s|}{|a_0 \wedge s|} \tag{3.5}$$

Then c is a direction, $\rho > 0$. Let $s' = -\epsilon s/|a_0 \wedge s|$. Then $s' = c - \rho a_0$. A point represented by a null vector p is on the hypersphere (c, ρ) if and only if $p \cdot s = 0$.

5. If $a_0 \cdot s = 0$, then a point represented by a null vector p is on the hyperplane normal to s if and only if $p \cdot s = 0$. ■

3.2. Relation Between Two Total Spheres

Definition 3.1. 1. Two hyperplanes are said to be parallel if their spheres at infinity have one and only one common point at infinity. They are said to be ultraparallel if they have a unique common perpendicular line.

2. Two spheres (or horospheres, or hyperspheres) are said to be concentric if their centers are collinear. Two hyperspheres are said to be tangent at infinity if they do not intersect and their axes are parallel.

3. Two hyperspheres are said to be same-sided if their axes are either identical or do not intersect, and for any line intersecting both hyperspheres and both axes at points p_1, p_2, q_1, q_2 on the same branch of $\mathscr{D}^n$, the order of p_1, q_1 is the same as the order of p_2, q_2 on the line.

Theorem 3.4. Let $s_1^{\sim}$, $s_2^{\sim}$ represent two distinct total spheres other than the sphere at infinity of $\mathscr{D}^n$.

1. If $a_0 \wedge s_1 \wedge s_2 = 0$, the two total spheres are concentric spheres, horospheres, or hyperspheres if and only if $a_0 \cdot (s_1 \wedge s_2)$ is negative-squared, null, or positive-squared, respectively.
2. If $s_1 \wedge s_2$ is Euclidean, $a_0 \wedge s_1 \wedge s_2 \neq 0$, and not both total spheres are hyperplanes, they intersect and the intersection is the generalized $(n-2)$-sphere $(s_1 \wedge s_2)^{\sim}$.
3. If $s_1 \wedge s_2$ is degenerate, $a_0 \wedge s_1 \wedge s_2 \neq 0$, and not both total spheres are hyperplanes, they are tangent to each other at the point or point at infinity corresponding to null vector $P_{s_1}(s_2)$.
4. If $s_1 \wedge s_2$ is Minkowski, $a_0 \wedge s_1 \wedge s_2 \neq 0$, and not both total spheres are hyperplanes, they do not intersect. There is a unique pair of points or a point and a point at infinity that are inversive with respect to both total spheres.
5. If both total spheres are hyperplanes, they intersect, are parallel, or ultraparallel if and only if $s_1 \wedge s_2$ is Euclidean, degenerate, or Minkowski, respectively.

The proof is based upon the fact that the intersection of the two spaces $s_1^{\sim}$, $s_2^{\sim}$ corresponds to the blade $s_1^{\sim} \vee s_2^{\sim} = (s_1 \wedge s_2)^{\sim}$, which is Minkowski, degenerate, or Euclidean if and only if the number of null 1-subspaces is 2, 1, or 0, respectively.

More specific conclusions can be established on the intersections and tangencies of pairs of total spheres. Below we present a theorem on pairs of hyperspheres.

Lemma 3.5. Let $c_1^{\sim}$, $c_2^{\sim}$ be two nonintersecting hyperplanes. Let p_1, p_2 be points on the same branch of $\mathscr{D}^n$ such that p_1 is on $c_1^{\sim}$ and p_2 is on $c_2^{\sim}$. Then $C = c_1 \cdot c_2 p_1 \cdot c_2 p_2 \cdot c_1 < 0$.

Proof. Since the hyperplanes do not intersect, $c_1 \cdot c_2 \neq 0$ and $p_1 \cdot c_2 \neq 0$ for any point p_1 on hyperplane $c_1^{\sim}$. Similarly, $p_2 \cdot c_1 \neq 0$ for any point p_2 on hyperplane $c_2^{\sim}$. So $C \neq 0$ and its sign depends on c_1, c_2 only.

When the hyperplanes are ultraparallel, let p_1p_2 be the common perpendicular line, p_1, p_2 be the intersections of the line with the two hyperplanes on the same branch of $\mathcal{D}^n$. Then

$$p_1 = \pm\frac{c_1\,(c_1 \wedge c_2)}{|c_1 \wedge c_2|}, \qquad p_2 = \pm\frac{c_1 \cdot c_2}{|c_1 \cdot c_2|}\,\frac{c_2\,(c_1 \wedge c_2)}{|c_1 \wedge c_2|} \tag{3.6}$$

$$C = -|c_1 \cdot c_2|\,|c_1 \wedge c_2| < 0 \tag{3.7}$$

When they are parallel, since C is a continuous function of its variables, $C \leq 0$. As $C \neq 0$, we get $C < 0$. ■

Lemma 3.6. Two hyperspheres (c_1, ρ_1) and (c_2, ρ_2) are same-sided if and only if $c_1 \cdot c_2 \geq 1$.

Proof. Let there be a line intersecting both hyperspheres and both axes at points p_1, p_2, q_1, q_2 on a branch of $\mathcal{D}^n$. Then

$$p_i \cdot c_i = -\rho_i, \qquad q_i \cdot c_i = 0, \qquad \text{for} \quad i = 1, 2 \tag{3.8}$$

If the two axes do not intersect, then $(c_1 \wedge c_2)^2 = (c_1 \cdot c_2)^2 - 1 \geq 0$, so $|c_1 \cdot c_2| \geq 1$. Since $q_1 \neq q_2$, $p_1 \wedge q_1 = \lambda q_1 \wedge q_2$. Making inner product on both sides with c_1 and applying (3.8), we get $\lambda = \rho_1/q_2 \cdot c_1$. Similarly, we have $p_2 \wedge q_2 = \mu q_1 \wedge q_2$, where $\mu = -\rho_2/q_1 \cdot c_2$. The pair p_1, q_1 have the same order as the pair p_2, q_2 on the line if and only if $p_1 \wedge q_1$ and $p_2 \wedge q_2$ have the same orientation, which is equivalent to

$$\lambda\mu = -\frac{\rho_1\,\rho_2}{q_2 \cdot c_1\; q_1 \cdot c_2} > 0 \tag{3.9}$$

By Lemma 3.5, $\lambda\mu/c_1 \cdot c_2 > 0$. So $\lambda\mu > 0$ if and only if $c_1 \cdot c_2 > 0$, or more accurately, $c_1 \cdot c_2 \geq 1$.

If the two axes intersect, then $|c_1 \cdot c_2| < 1$, and $c_1 \cdot c_2 \geq 1$ is not satisfied.

If the two axes are identical, then $q_1 = q_2 = q$, $c_1 = \epsilon c_2$, where $\epsilon = \pm 1$. In particular, $|c_1 \cdot c_2| = 1$. Let a be a tangent vector of the line at point q; then $a \cdot q = 0$ and $a \wedge q \neq 0$. Similar to the nonintersecting case, we get

$$p_1 \wedge q = -\frac{\rho_1}{\epsilon a \cdot c_2}\, a \wedge q, \qquad p_2 \wedge q = -\frac{\rho_2}{a \cdot c_2}\, a \wedge q \tag{3.10}$$

$p_1 \wedge q_1$ and $p_2 \wedge q_2$ have the same orientation if and only if $\epsilon = 1$, or equivalently, $c_1 \cdot c_2 = 1$. ■

Theorem 3.7. Let $s_1^{\sim}$, $s_2^{\sim}$ represent two distinct hyperspheres.

1. If the two hyperspheres intersect, then (a) if their axes are ultraparallel, the intersection is an $(n-2)$-sphere; (b) if their axes are parallel, the intersection is an $(n-2)$-horosphere; and (c) if their axes intersect, the intersection is an $(n-2)$-hypersphere. The center and radius of the intersection are the same as those of the generalized sphere $(P_{s_1\wedge s_2}(a_0))^{\sim}$.
2. If the two hyperspheres are tangent, their axes are parallel or ultraparallel. The tangency occurs at infinity if and only if the hyperspheres have parallel axes and equal radii, and are same-sided.
3. If the two hyperspheres neither intersect nor are tangent, their axes are ultraparallel.

Proof. Let $s_i = c_i - \rho_i a_0$ for $i = 1, 2$, where $c_i \cdot a_0 = 0$, $c_i^2 = 1$, and $\rho_i > 0$.

1. When the two hyperspheres intersect, then $(s_1 \wedge s_2)^2 < 0$. The Minkowski blade $(P_{s_1\wedge s_2}(a_0))^{\sim}$ represents a generalized sphere, since neither $a_0 \cdot (P_{s_1\wedge s_2}(a_0))^{\sim}$ nor $a_0 \wedge (P_{s_1\wedge s_2}(a_0))^{\sim}$ equals zero. Using the formula

$$(s_1 \wedge s_2)^{\sim} = \frac{(s_1 \wedge s_2)^2}{(a_0 \cdot (s_1 \wedge s_2))^2}\,(a_0 \cdot (s_1 \wedge s_2))(P_{s_1\wedge s_2}(a_0))^{\sim} \tag{3.11}$$

we get that the total $(n-2)$-sphere $(s_1 \wedge s_2)^{\sim}$ is the intersection of the hyperplane $(a_0 \cdot (s_1 \wedge s_2))^{\sim}$ with the generalized sphere $(P_{s_1\wedge s_2}(a_0))_{\sim}$. Since

$$P_{a_0}^{\perp}(P_{s_1\wedge s_2}(a_0)) \cdot (a_0 \cdot (s_1 \wedge s_2)) = P_{a_0}(P_{s_1\wedge s_2}(a_0)) \cdot (a_0 \cdot (s_1 \wedge s_2)) = 0 \tag{3.12}$$

the center of the generalized sphere is in the hyperplane. Therefore, $(s_1 \wedge s_2)^{\sim}$ is a generalized $(n-2)$-sphere whose center and radius are the same as those of $(P_{s_1\wedge s_2}(a_0))^{\sim}$. The intersection is an $(n-2)$-dimensional sphere, horosphere, or hypersphere if the blade $a_0 \cdot (P_{s_1\wedge s_2}(a_0))^{\sim} = (a_0 \wedge P_{s_1\wedge s_2}(a_0))^{\sim}$ is Euclidean, degenerate, or Minkowski, respectively. Using

$$(s_1 \wedge s_2)^4 (a_0 \wedge P_{s_1\wedge s_2}(a_0))^2 = (c_1 \wedge c_2)^2 (\rho_2 c_1 - \rho_1 c_2)^2 \tag{3.13}$$

$$(s_1 \wedge s_2)^2 = (c_1 \wedge c_2)^2 - (\rho_2 c_1 - \rho_1 c_2)^2 \tag{3.14}$$

we get that $(a_0 \wedge P_{s_1\wedge s_2}(a_0))^2$ has the same sign as $(c_1 \wedge c_2)^2$. From this and Theorem 3.4, we get the conclusions on the intersection.

2. When the two hyperspheres are tangent, then $(s_1 \wedge s_2)^2 = 0$. If the axes intersect, then $c_1 \wedge c_2$ is Euclidean, so $\rho_2 c_1 - \rho_1 c_2$ has positive square. As a result, $(s_1 \wedge s_2)^2 < 0$ by (3.14), which is a contradiction.

The tangency occurs at infinity if $(s_1 \wedge s_2)^2 = (c_1 \wedge c_2)^2 = 0$, which is equivalent to $c_1 \cdot c_2 = 1$ and $\rho_1 = \rho_2$.

3. When the two hyperspheres neither intersect nor are tangent, then $(s_1 \wedge s_2)^2 > 0$. By (3.14), $(\rho_2 c_1 - \rho_1 c_2)^2 < (c_1 \wedge c_2)^2$. If $c_1 \wedge c_2$ is not Minkowski, then $(c_1 \wedge c_2)^2 \leq 0$; on the other hand, $\rho_2 c_1 - \rho_1 c_2$ has a nonnegative square because it is a vector in $c_1 \wedge c_2$. As a result, $(\rho_2 c_1 - \rho_1 c_2)^2 \geq (c_1 \wedge c_2)^2$, which is a contradiction. ■

3.3. Bundles of Total Spheres

The main content in the previous subsection is a special case of the content in this subsection.

Definition 3.2. A bundle of total spheres determined by r total spheres which are represented by Minkowski $(n + 1)$-blades $B_1, \ldots, B_r$ is the set of total spheres represented by $\lambda_1 B_1 + \ldots + \lambda_r B_r$, where the λ's are scalars.

When $B_1 \vee \ldots \vee B_r \neq 0$, the integer $r - 1$ is called the dimension of the bundle. A one-dimensional bundle is called a pencil. The dimension of a bundle is allowed to be between 1 and $n - 1$. The blade $A_{n-r+2} = B_1 \vee \ldots \vee B_r$ can be used to represent the bundle. There are five classes:

1. When $a_0 \cdot A_{n-r+2} = 0$, the bundle is called a concentric bundle. It is composed of the sphere at infinity and the generalized spheres whose centers lie in the subspace $(a_0 \wedge A_{n-r+2})^{\sim}$ of $\mathcal{R}^{n,1}$.
2. When A_{n-r+2} is Minkowski and $a_0 \cdot A_{n-r+2}, a_0 \wedge A_{n-r+2} \neq 0$, the bundle is called a concurrent bundle. It is composed of total spheres containing the generalized $(n - r)$-sphere A_{n-r+2}. In particular, when A_{n-r+2} represents an $(n - r)$-hypersphere, the bundle is composed of hyperspheres only. $a_0 \wedge (a_0 \cdot A_{n-r+2})$ represents the axis of the $(n - r)$-hypersphere and is the intersection of all axes of the hyperspheres in the bundle.
3. When A_{n-r+2} is degenerate and $a_0 \cdot A_{n-r+2}, a_0 \wedge A_{n-r+2} \neq 0$, the bundle is called a tangent bundle. Any two nonintersecting total spheres in the bundle are tangent to each other. The tangency occurs at the point or point at infinity corresponding to the unique null 1-space in the space A_{n-r+2}.
4. When A_{n-r+2} is Euclidean and $a_0 \cdot A_{n-r+2}, a_0 \wedge A_{n-r+2} \neq 0$, the bundle is called a Poncelet bundle. $A^{\sim}_{n-r+2}$ represents a generalized $(r - 2)$-sphere, which is self-inversive with respect to every total sphere in the bundle.
5. When $a_0 \wedge A_{n-r+2} = 0$, the bundle is called a hyperplane bundle. It is composed of hyperplanes (1) perpendicular to the $(r - 1)$-plane represented by $a_0 \wedge A^{\sim}_{n-r+2}$, or (2) whose representations in the homogeneous model contain the blade A_{n-r+2}, or (3) passing through the $(n - r)$-plane represented by A_{n-r+2} if the blade

A_{n-r+2} is (1) Euclidean, or (2) degenerate, or (3) Minkowski, respectively.

4. CONFORMAL TRANSFORMATIONS

It is a well-known fact that the orthogonal group $O(n+1, 1)$ is a double covering of the conformal group $M(n)$ of $\mathcal{D}^n$: $M(n) = O(n+1, 1)/\langle \pm 1 \rangle$.

In Clifford algebra, $O(n+1,1)$ is isomorphic to the projective pin group $Pin(n+1, 1)$, which is the quotient of the versor group of $\mathcal{R}^{n+1,1}$ by $\mathcal{R} - \{0\}$. The group $Pin(n+1, 1)$ has four connected components:

- $E_+(n+1, 1)$, the set of versors which are geometric products of an even number of positive-squared vectors and an even number of negative-squared vectors. It is a subgroup of $Pin(n+1, 1)$, which is isomorphic to the proper Lorentz group $Lor^+(n+1)$.
- $E_-(n+1, 1)$, the set of versors which are geometric products of an odd number of positive-squared vectors and an odd number of negative-squared vectors. The sets $E_-(n+1, 1)$ and $E_+(n+1, 1)$ form a subgroup of $Pin(n+1, 1)$, which is isomorphic to the special orthogonal group $SO(n+1, 1)$.
- $O_+(n+1, 1)$, the set of versors which are geometric products of an odd number of positive-squared vectors and an even number of negative-squared vectors. The sets $O_+(n+1, 1)$ and $E_+(n+1, 1)$ form a subgroup of $Pin(n+1, 1)$, which is isomorphic to the Lorentz group $Lor(n+1)$.
- $O_-(n+1, 1)$, the set of versors which are geometric products of an even number of positive-squared vectors and an odd number of negative-squared vectors. The sets $O_-(n+1, 1)$ and $E_+(n+1, 1)$ form a subgroup of $Pin(n+1, 1)$, which is isomorphic to the skew Lorentz group $Lor^-(n+1) = \{ f \in O(n+1, 1) \mid f(\mathcal{H}^n) = \det(f)\mathcal{H}^n \}$.

Let $I_{n+1,1}$ be a unit pseudoscalar. Then the versor action of $I_{n+1,1}$ maps x to $-x$ for any x in $\mathcal{R}^{n+1,1}$. Therefore,

$$M(n) = Pin(n+1, 1)/\{I_{n+1,1}\} \tag{4.15}$$

which serves as the second main theorem on the homogeneous model:

Theorem 4.1. Any conformal transformation in $\mathcal{D}^n$ can be realized in the homogeneous model by the versor action of a versor in $\mathcal{G}_{n+1,1}$, and vice versa. Any two versors realize the same conformal transformation if and only if they are equal up to a nonzero scalar or pseudoscalar factor.

The paeudoscalar $I_{n+1,1}$ induces a duality in $\mathcal{G}_{n+1,1}$ under which $O_+(n + 1, 1)$ and $O_-(n + 1, 1)$ are interchanged when n is even, and $O_+(n + 1, 1)$ and $E_-(n + 1, 1)$ are interchanged when n is odd. So,

$$M(n) = \begin{cases} Lor(n + 1) = Lor^-(n + 1) & \text{when } n \text{ is even} \\ Lor(n + 1) = SO(n + 1,1) & \text{when } n \text{ is odd} \end{cases} \tag{4.16}$$

Now we use the versor representation to study a conformal transformation which is similar to dilation in Euclidean space. This is the tidal transformation, whose corresponding versor is $1 + \lambda a_0 c$, where $\lambda \in \mathcal{R}$, $c \in \mathcal{R}^{n+1,1}$ and $c \cdot a_0 = 0$.

Under this transformation, the concentric pencil $(a_0 \wedge c)^{\sim}$ is invariant. When c is a point or end, the set $\{c, -c\}$ is invariant; when c is a direction, the hyperplane $c^{\sim}$ is not invariant, while its sphere at infinity is invariant.

Assume that p is a fixed point in $\mathcal{D}^n$ and is transformed to a point or point at infinity q by a tidal transformation with parameter λ. It can be proved that λ is a monotonous function of q on line $c \wedge p$ except a point or point at infinity. Below we list some results on $\lambda = \lambda(q)$.

1. When c is a point (Fig. 1):
   (a) For any point or point at infinity q on line $c \wedge p$, let $C_c(q) = -c^{-1}qc$; then $\lambda(-C_c(q)) = 1/\lambda(q)$.
   (b) For any point q on line $c \wedge p$, $\lambda(q) = (q - p)^2/[(q - c)^2 - (p - c)^2]$.
   (c) $\lambda(p - e^{\pm d(p,c)}c) = e^{\pm d(p,c)}$.

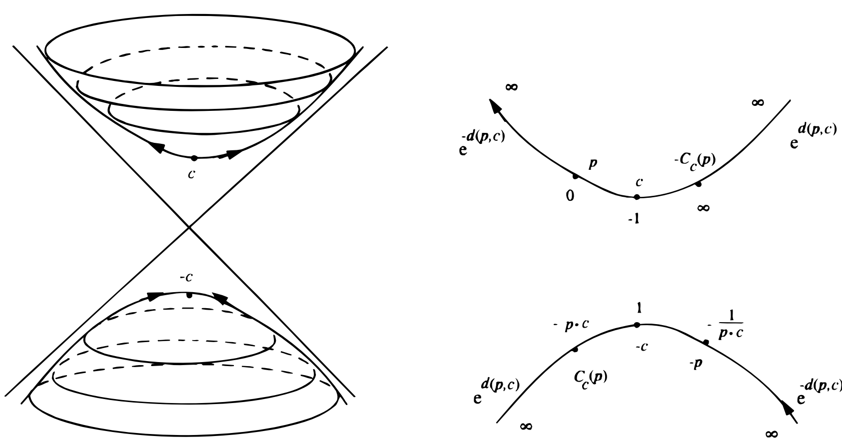

Fig. 1. Tidal transformation when c is a point. Right: $\lambda = \lambda(q)$. The hyperbola represents the line $c \wedge p$. The values of λ are between the two branches of the hyperbola for the corresponding values of q, which are outside. The arrows on the hyperbola indicate the direction of increasing λ.

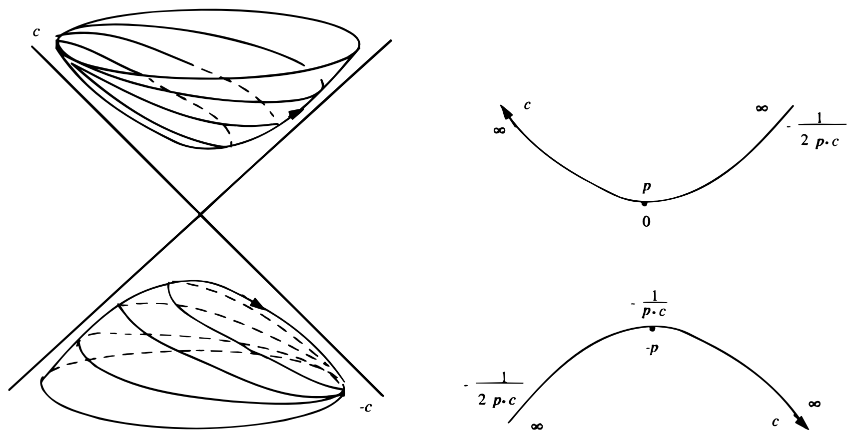

Fig. 2. Tidal transformation when c is an end. Right: $\lambda = \lambda(q)$.

2. When c is an end (Fig. 2):
   (a) For any point q on line $c \wedge p$, $\lambda(q) = \frac{1}{2}(1/q \cdot c - 1/p \cdot c)$.
   (b) If q is the end of line $c \wedge p$ other than c, then $\lambda(g) = -1/(2p \cdot c)$.
3. When c is a direction (Fig. 3):
   (a) For any point or point at infinity q on line $c \wedge p$, $\lambda(-C_c(q)) = -1/\lambda(g)$.
   (b) For any point q on line $c \wedge p$, $\lambda(q) = (q - p)^2/[(q - c)^2 - (p - c)^2]$.
   (c) Assume $p \cdot c < 0$, and let $d(p, c)$ be the hyperbolic distance from p to the intersection t of line $c \wedge p$ with hyperplane $c^{\sim}$ on the branch of $\mathscr{D}^n$ containing p; then

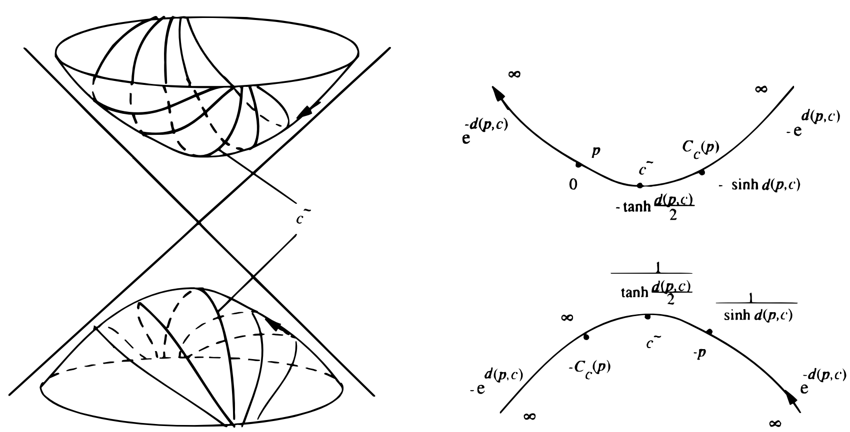

Fig. 3. Tidal transformation when c is a direction. Right: $\lambda = \lambda(q)$.

$$\lambda(C_c(p)) = -\sinh d(p, c), \qquad \lambda(t) = -\tanh \frac{d(p, c)}{2}$$

$$\lambda(p + e^{d(p,c)}c) = -e^{d(p,c)}, \qquad \lambda(p + e^{-d(p,c)}c) = e^{-d(p,c)}$$

(d) Assume that $p \cdot c = 0$ and that q is on the branch of $\mathcal{D}^n$ containing p; then $\lambda(q) = -\epsilon \tanh[d(p, q)/2]$, where ϵ is the sign of $q \cdot c$.

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